Journal of Engineering Mathematics 18 (1984) 105–117

Power-law creep of a material being compressed between parallel plates: a singular perturbation problem

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(Received July 1, 1983 and in revised form December 12, 1983)

Summary

The flow of a power-law creeping or viscous material which is being compressed in a narrow gap between parallel plates is studied. A perturbation scheme based on the small gap size is developed and the approximations which lead to classical lubrication theory are formally identified. The solution obtained from lubrication theory is shown to correspond to an outer solution which is not uniformly valid because it predicts infinite longitudinal stresses along both the centerline of the gap and across the entire gap on the line connecting the plate midpoints. The failure of lubrication theory to describe the flow in these regions is not due to an inherent failure of the power-law constitutive equation to model material behavior, but is due to a breakdown in the approximations made in lubrication theory. The solution is corrected by constructing inner solutions in the regions where lubrication theory fails and a uniformly valid solution for the stress field and velocity field is obtained.

1. Introduction

The present study is a detailed analysis of the flow of a power-law creeping material which is being squeezed between parallel plates. Scott [1] was the first to examine this squeezing-flow problem. He was interested in deducing material properties from readings obtained from a parallel-plate plastimeter. A summary of Scott's analysis may be found in many papers; for example, Leider and Bird [2], and Brindley, Davies and Walter [3]. Previous analyses of this problem have used the approximations of classical lubrication theory from the outset, based on heuristic arguments concerning the smallness of the gap between the plates and neglecting inertial effects. Furthermore, in the majority of the work the goal has generally been to obtain rather global results, such as an expression for the net force on the plates in terms of the plate velocity and material properties. Here we present a detailed analysis of the flow by formally developing the solution in terms of a perturbation expansion based on the small gap size. This formal treatment reveals that, in the case of a power-law fluid, the lubrication approximation fails to be uniformly valid and inner solutions are required near the gap centerline and near the line connecting the plate midpoints. Flow details in these regions are relevant to the micromechanical analysis of void growth and suspended particle transport.

For the power-law creeping material considered here the effective viscosity is proportional to the inverse of a power of the second stress invariant. In the lubrication theory the second stress invariant is approximated by the square of the shear stress. Consequently, along the centerline of the gap, where the shear stress vanishes by symmetry, the effective viscosity predicted by the lubrication approximation is infinite. Therefore, since the strain rate is nonzero there, the lubrication theory predicts infinite stress at the centerline of the gap. This physically unacceptable singularity is not inherent in the constitutive equation for a power-law creeping material but is simply a failure of the lubrication approximation. In particular, the lubrication approximation is not uniformly valid and an inner solution must be constructed in the neighborhood of the centerline.

For similar reasons the lubrication approximation also fails across the entire gap in the vicinity of the line which connects the plate midpoints. In this case the failure is not only due to the fact that the shear stress vanishes (by symmetry), but is also due to a breakdown in the scaling for the velocity field. As we shall see, the lubrication approximation assumes that the component of velocity parallel to the plates is large compared to the perpendicular component. This assumption is incorrect in the plate midpoint region, since the velocity components are expected to be of the same order of magnitude there.

We shall find that the inner solutions discussed above are necessary in order to obtain a uniformly valid description of the stress field, but they do not significantly affect the velocity field predicted by the outer or lubrication-theory solution.

The goal of the present paper is primarily aimed at resolving a somewhat mathematical question. Namely, we determine the precise structure of the solution for the compressive flow of a power-law viscous material and demonstrate that a solution free from any undesirable singular behavior may be obtained by using the method of matched asymptotic expansions. Nevertheless, some discussion of the practical value of the present problem is appropriate here. This requires an examination of the usefulness and limitations of the constitutive equation being considered.

The materials for which the power-law model is useful include metals at high temperature (generally temperatures greater than half of the melting temperature) and some polymeric liquids. At temperatures above about half the melting temperature and strain rates greater than 10^{-6} sec^{-1} many metals are modeled reasonably well by the power-law equation considered here [4,5,6]. The deformation mechanism in this case is generally referred to as dislocation creep. For strain rates below 10^{-6} sec^{-1}, however, diffusional creep dominates and the material behaves in a linearly viscous fashion. Due to this linearly viscous behavior at small strain rates the power-law constitutive equation becomes a poor model of the material in the same regions where the lubrication approximation fails, i.e., in regions where the shear stress or shear strain rate becomes small. However, the constitutive equation generally fails at such small values of the strain rate that the size of the inner regions which correct the lubrication theory can often be much larger than the size of the regions where the power-law equation fails. In particular, in the present analysis we find that we enter the inner regions when the shear strain rate is of the order V/h where V is the plate velocity and h is the gap width. Consequently, whenever this quantity is greater than 10^{-6} sec^{-1}, (i.e., the strain rate below which we typically expect linearly viscous behavior) the present analysis is relevant since the regions where the constitutive equation fails would be small and embedded within the inner regions which are examined here. We should note that the linearly viscous behavior at very small stresses could be modeled using a generalized constitutive equation which includes an additive constant in the effective viscosity, however, such a generalization here only complicates the analysis and does not serve our primary purpose.

Similarly, in the case of polymeric liquids a power-law model may also be an adequate description. However, although polymeric liquids are generally shear thinning many polymeric liquids show tensile thickening behavior which is in contrast to the tensile

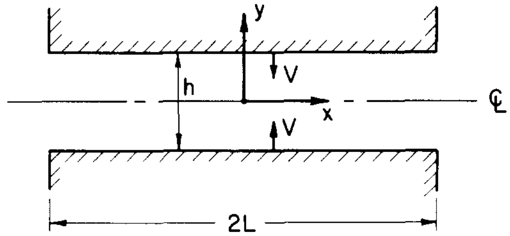

Figure 1. Squeeze film geometry.

thinning behavior being considered here. Nevertheless, there have been a few observations of liquids which have an elongational viscosity which decreases with increasing stress or elongation rate [7,8]. Given the fact that polymeric liquids are typically shear thinning, that some of the data also show tensile thinning behavior, and the likelihood that future data will reveal other tensile thinning liquids, it seems appropriate to note that the present analysis may be relevant to some polymeric liquids. Note also that the previous discussion of the linearly viscous region is relevant here since polymeric liquids behave in a linearly viscous manner for strain rates below about 10^{-1} $\sec^{-1}$.

The paper begins with a rigorous formulation of the solution of classical lubrication theory which in the context of singular-perturbation theory corresponds to the outer solution. This solution is then examined in order to understand why it fails, and then solutions for the inner regions are developed. Lastly, note that in the gravity driven flow of a glacier Johnson and McMeeking [9] analyzed a surface layer at the glacier's surface which is similar in character to the inner solutions studied here.

2. Formulation

We consider the inertialess flow of a power-law creeping material being compressed between plane parallel plates (Fig. 1). The separation of the plates h is assumed small compared to the length $2L$ of the plates, and the plates move toward one another at a velocity V. The $\hat{x}$-axis is taken to lie along the gap centerline with $\hat{x} = 0$ being coincident with the midpoint of the plates. The $\hat{y}$-axis measures the distance from the centerline with $\hat{y} = \pm h/2$ being the position of the plates. We consider steady motion in the $\hat{x}$, $\hat{y}$-plane only, i.e., plane strain.

The material filling the gap is incompressible and obeys the constitutive equation

$$\hat{e}_{ij} = B\hat{\tau}^{n-1}\hat{s}_{ij},$$

where $\hat{\tau}^2 = 1/2\hat{s}_{ij}\hat{s}_{ij}$, $\hat{e}_{ij}$ is the strain rate, $\hat{s}_{ij}$ is the deviatoric stress, and B and n are material parameters ($n \geqslant 1$). We introduce the nondimensional variables $x = \hat{x}/L, y = \hat{y}/h$, $\boldsymbol{u} = (u, v) = \hat{\boldsymbol{u}}/V$, $\tau_{ij} = \hat{\tau}_{ij}/\tau_0$, $s_{ij} = \hat{s}_{ij}/\tau_0$, where $\tau_{ij} = 1/2\tau_{kk}\delta_{ij} + s_{ij}$ and $\tau_0 = [VL/Bh^2]^{1/n}$. The carot above a variable indicates the physical quantity. Here u and v are the components of the velocity field in the x- and y-directions respectively. With the restriction that the gap is small we require $\epsilon \equiv h/L \ll 1$ and in terms of the nondimensional variables the governing equations become

$$\epsilon \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial s_{xy}}{\partial y} = 0, \qquad \epsilon \frac{\partial s_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = 0, \tag{1}$$

$$\epsilon \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2}$$

For the constitutive equations we have

$$\epsilon^2 \frac{\partial u}{\partial x} = \tau^{n-1} s_{xx}, \qquad \epsilon \frac{\partial v}{\partial y} = \tau^{n-1} s_{yy}, \tag{3}$$

$$\epsilon \frac{\partial u}{\partial y} + \epsilon^2 \frac{\partial v}{\partial x} = 2\tau^{n-1} s_{xy}. \tag{4}$$

Note that (3a) and (3b) are equivalent since $s_{xx} = -s_{yy}$ and from (2) $\partial v/\partial y = -\epsilon \partial u/\partial x$.

The boundary conditions for the problem are: the no-slip condition on the plates,

$$u = 0, \qquad v = -1, \qquad \text{at} \quad y = 1/2, \tag{5}$$

and the symmetry conditions at the centerline and at the line connecting the plate midpoints,

$$\frac{\partial u}{\partial y} = 0, \qquad v = 0, \qquad \text{at} \quad y = 0, \tag{6}$$

$$u = 0, \qquad \frac{\partial \tau_{xx}}{\partial x} = 0 \quad \text{or} \quad s_{xy} = 0, \qquad \text{at} \quad x = 0. \tag{7}$$

Note that we consider flow which is symmetric about the geometric lines of symmetry since this corresponds to the situation which is realized in laboratory experiments.

In addition, we require a boundary condition at the exit of the gap. It is sufficient for the present purposes to take the net longitudinal force at the exit equal to zero, i.e.,

$$\int_{-1/2}^{1/2} \tau_{xx} \mathrm{d}y = 0 \qquad \text{at} \quad x = 1. \tag{8}$$

Note that a more general version of the problem would not specify a condition at $x = 1$, but would continue the solution into the region outside of the gap where on the free-surface of the film the normal and tangential components of the stress would be required to vanish (neglecting surface tension) and we would impose the kinematic boundary condition on the velocity. However, solving this problem in the neighborhood of the exit is difficult and requires substantial numerical computation. Since our interest is primarily concerned with the flow in the gap we shall not consider these details of the flow, but use the integral condition (8) which is a necessary constraint in order to have force equilibrium for the film outside of the gap. Further discussion of this boundary condition is given in [10].

In the outer region we shall denote the components of the stress and velocity field by τ_{xx}, τ_{yy}, s_{xy}, u and v. Within the inner regions we shall denote the corresponding field quantities by the following upper-case letters: T_{xx}, T_{yy}, S_{xy}, U and V.

3. Outer solution: Classical lubrication theory

The continuity equation (2) suggests the following expansion for the velocity

$$u=\epsilon^{-1}\left[u^{(0)}+\epsilon u^{(1)}+\dots\right], \tag{9}$$

$$v=v^{(0)}+\epsilon v^{(1)}+\dots. \tag{10}$$

Here we see that the velocity parallel to the plates is generally large compared to the velocity perpendicular to the plates. From the equilibrium and constitutive equations we are led to expand the stresses as

$$\tau_{kk}=\epsilon^{-1}\tau_{kk}^{(-1)}+\tau_{kk}^{(0)}+\epsilon\tau_{kk}^{(1)}+\dots, \qquad (k=x \text{ or } y, \text{ no sum}) \tag{11}$$

$$s_{ij}=s_{ij}^{(0)}+\epsilon s_{ij}^{(1)}+\dots. \tag{12}$$

Consequently, $\tau_{xx}^{(-1)}=\tau_{yy}^{(-1)}$, and τ^{n-1} which appears in the constitutive equation is given by

$$\tau^{n-1}=\tau_0^{n-1}+\epsilon\tau_1^{n-1}+\dots, \tag{13}$$

$$\tau_0^{n-1}=\left[s_{xx}^{(0)^2}+s_{xy}^{(0)^2}\right]^{(n-1)/2}, \tag{14}$$

$$\tau_1^{n-1}=(n-1)\left(s_{xx}^{(1)}s_{xx}^{(0)}+s_{xy}^{(1)}s_{xy}^{(0)}\right)\tau_0^{n-3}. \tag{15}$$

After substituting the expansions into the governing equations, we have at leading-order

$$\frac{\partial\tau_{yy}^{(-1)}}{\partial y}=0, \qquad \tau_{xx}^{(-1)}=\tau_{yy}^{(-1)}. \tag{16}$$

Therefore $\tau_{xx}^{(-1)}$ and $\tau_{yy}^{(-1)}$ are functions only of x and we define $\partial\tau_{xx}^{(-1)}/\partial x\equiv G_{-1}(x)$.

At next order the equilibrium equations give

$$\frac{\partial s_{xy}^{(0)}}{\partial y}=-\frac{\partial\tau_{xx}^{(-1)}}{\partial x}=-G_{-1}(x), \qquad \frac{\partial\tau_{yy}^{(0)}}{\partial y}=0, \tag{17}$$

$$\frac{\partial u^{(0)}}{\partial x}+\frac{\partial v^{(0)}}{\partial y}=0, \tag{18}$$

and the constitutive equations give

$$s_{xx}^{(0)}=-s_{yy}^{(0)}=0, \qquad \frac{\partial u^{(0)}}{\partial y}=2\tau_0^{n-1}s_{xy}^{(0)}. \tag{19}$$

Note that using (19a) in (14) gives $\tau_0^{n-1} = |s_{xy}^{(0)}|^{n-1}$. This is one of the fundamental assumptions generally taken at the outset in classical lubrication theory. Furthermore, it is this approximation which ultimately leads to the breakdown of the outer solution.

From (17) and the symmetry condition at $y = 0$ we have

$$s_{xy}^{(0)} = -G_{-1}(x)y. \tag{20}$$

Equation (17b) shows that $\tau_{yy}^{(0)}$ depends only on x and from (19a) we find that $\tau_{xx}^{(0)} = \tau_{yy}^{(0)}$. Therefore, for convenience we define $\partial\tau_{xx}^{(0)}/\partial x \equiv G_0(x)$.

Lastly, from (18), (19), (20) and the no-slip condition $u = 0$ on $y = 1/2$ we obtain

$$u^{(0)} = -\frac{2}{n+1}G_{-1}^{n}\left[y^{n+1} - (1/2)^{n+1}\right], \tag{21}$$

$$v^{(0)} = 2\frac{n}{n+1}G_{-1}^{n-1}\frac{\partial G_{-1}}{\partial x}\left[\frac{y^{n+1}}{n+2} - (1/2)^{n+1}\right]y. \tag{22}$$

The boundary condition $v^{(0)} = -1$ on $y = 1/2$ and the symmetry condition at $x = 0$ gives

$$G_{-1}(x) = 2^{1+1/n}(n+2)^{1/n}x^{1/n} = -\frac{\partial\tau_{xx}^{(-1)}}{\partial x}. \tag{23}$$

Consequently, after using the boundary condition (8) at $x = 1$, the leading order longitudinal stress is found to be

$$\tau_{xx}^{(-1)} = \tau_{yy}^{(-1)} = 2^{1+1/n}(n+2)^{1/n}\frac{n}{1+n}\left(x^{(1+n)/n} - 1\right). \tag{24}$$

For the velocity field we finally find

$$u^{(0)} = -\frac{n+2}{n+1}2^{n+2}x\left[y^{n+1} - (1/2)^{n+1}\right], \tag{25}$$

$$v^{(0)} = \frac{n+2}{n+1}2^{n+2}\left[\frac{y^{n+1}}{n+2} - (1/2)^{n+1}\right]y. \tag{26}$$

At $O(\epsilon)$ we have the governing equations

$$\frac{\partial\tau_{yy}^{(1)}}{\partial y} = -\frac{\partial s_{xy}^{(0)}}{\partial x} = y\frac{\partial G_{-1}}{\partial x}, \qquad \frac{\partial s_{xy}^{(1)}}{\partial y} = -\frac{\partial\tau_{xx}^{(0)}}{\partial x} \equiv -G_0(x), \tag{27}$$

$$\frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} = 0, \tag{28}$$

and the constitutive equations become

$$\tau_1^{n-1}s_{xx}^{(0)} + \tau_0^{n-1}s_{xx}^{(1)} = \frac{\partial u^{(0)}}{\partial x}, \tag{29}$$

$$\frac{\partial u^{(1)}}{\partial y} = 2\left(\tau_1^{n-1}s_{xy}^{(0)} + \tau_0^{n-1}s_{xy}^{(1)}\right). \tag{30}$$

Integrating (27) and using the fact that $s_{xx}^{(0)} = 0$ in (29) and (30) we obtain

$$\tau_{yy}^{(1)} = \frac{1}{2}\frac{\partial G_{-1}}{\partial x} y^2 + a(x), \qquad s_{xy}^{(1)} = -G_0 y, \tag{31}$$

$$\frac{\partial u^{(1)}}{\partial y} = 2n|s_{xy}^{(0)}|^{n-1} s_{xy}^{(1)}, \tag{32}$$

$$s_{xx}^{(1)} = \frac{1}{|s_{xy}^{(0)}|^{n-1}} \frac{\partial u^{(0)}}{\partial x}, \tag{33}$$

where, from (20) and (23), $s_{xy}^{(0)} = -G_{-1} y = -2^{1+1/n}(n+2)^{1/n} x^{1/n} y$. In (31) $a(x)$ is an integration factor which would ultimately be determined at next order, i.e., by continuing the solution through terms of $O(\varepsilon^2)$ in s_{xy}. We will not, however, pursue the solution to this order here.

Equations (31b), (32) and (28) give after using the no-slip boundary condition

$$u^{(1)} = -2\frac{n}{n+1} G_{-1}^{n-1} G_0 \left[y^{n+1} - (1/2)^{n+1} \right], \tag{34}$$

$$v^{(1)} = -2\alpha^{n-1} \frac{n}{n+1} x^{-1/n} y \left[\frac{y^{n+1}}{n+2} - (1/2)^{n+1} \right] \left[x\frac{\partial G_0}{\partial x} + \frac{n-1}{n} G_0 \right]. \tag{35}$$

where $\alpha = 2^{1+1/n}(n+2)^{1/n}$. From the remaining boundary condition on $v^{(1)}$, namely, $v^{(1)} = 0$ at $y = 1/2$, we require

$$x\frac{\partial G_0}{\partial x} + \frac{n-1}{n} G_0 = 0, \tag{36}$$

and therefore $G_0 = Cx^{(1/n)-1}$ where C is an integration constant which is determined from the matching with the inner solution near $x = 0$.

Finally from (33) and (25) we find

$$s_{xx}^{(1)} = -2^{2+1/n} \frac{(n+2)^{1/n}}{n+1} \frac{1}{x^{1-1/n} y^{n-1}} \left[y^{n+1} - (1/2)^{n+1} \right], \tag{37}$$

and $\tau_{xx}^{(1)}$ is given by $\tau_{xx}^{(1)} = \tau_{yy}^{(1)} + 2s_{xx}^{(1)}$ where $\tau_{yy}^{(1)}$ is given by (31).

Equation (37) shows that when either x or y approach zero $s_{xx}^{(1)}$ becomes unbounded and therefore the outer solution fails. This difficulty is due to the approximation of the effective viscosity $\tau_0^{1-n} \simeq [s_{xy}^{(0)}]^{1-n}$ which tends to infinity as x or y tend to zero since $s_{xy}^{(0)} \to 0$. A closer examination reveals that the expansion scheme used for τ fails to be a valid approximation when either x or y tend to zero. Consider the expansion for τ^2 valid in the outer region where we have $s_{xx}^{(0)} = 0$,

$$\tau^2 \simeq s_{xy}^{(0)^2} + 2\epsilon s_{xy}^{(1)} s_{xy}^{(0)} + \epsilon^2 \left[s_{xy}^{(1)^2} + s_{xx}^{(1)^2} + 2s_{xy}^{(0)} s_{xy}^{(2)} \right] + \dots.$$

As $y \to 0$, we have $\epsilon^2 s_{xx}^{(1)^2} = O[\epsilon^2/y^{2(n-1)}]$ in the last term and $s_{xy}^{(0)^2} = O[y^2]$ in the first term and therefore these two terms are of the same order of magnitude when $y = O(\epsilon^{1/n})$

(note that the second term is of $O(\epsilon)$ as $y \to 0$ and therefore not important here). Consequently, within a distance of $O(\epsilon^{1/n})$ about the centerline of the gap the expansion of the outer solution fails and an inner solution must be constructed. Similarly as $x \to 0$, $s_{xy}^{(0)^2} = O[x^{2/n}]$, $\epsilon s_{xy}^{(1)} s_{xy}^{(0)} = O[\epsilon / x^{1-2/n}]$ and $\epsilon^2 s_{xx}^{(1)^2} = O[\epsilon^2 / x^{2-2/n}]$ and therefore the expansion scheme breaks down when $x = O(\epsilon)$. In this case the breakdown of the outer solution is also evident from an examination of the continuity equation and the expansions used for the velocity field.

4. Centerline inner solution

Since the outer solution breaks down within a distance of order $\epsilon^{1/n}$ of the centerline, we introduce the inner variable $Y = y/\epsilon^{1/n}$. Using T_{xx}, T_{yy}, S_{xy} U and V to denote the solution in this region, the governing equations become

$$\epsilon^{1+1/n} \frac{\partial T_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial Y} = 0, \qquad \epsilon^{1+1/n} \frac{\partial S_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial Y} = 0, \tag{38}$$

$$\epsilon^{1+1/n} \frac{\partial U}{\partial x} + \frac{\partial V}{\partial Y} = 0, \tag{39}$$

$$\epsilon^2 \frac{\partial U}{\partial x} = -\tau^{n-1} S_{xx}, \qquad \epsilon^{1-1/n} \frac{\partial U}{\partial Y} + \epsilon^2 \frac{\partial V}{\partial x} = 2\tau^{n-1} S_{xy}. \tag{40}$$

Examining these equations and the inner limit of the outer solution motivates the following expansions of the solution in the centerline region

$$T_{kk} = \epsilon^{-1} T_{kk}^{(-1)} + \epsilon^{1/n} T_{kk}^{(0)} + \dots, \qquad (k = x \text{ or } y, \text{ no sum})$$

$$S_{ij} = \epsilon^{1/n} S_{ij}^{(0)} + \dots, \tag{41}$$

$$U = \epsilon^{-1} \left[U^{(0)} + \dots \right], \qquad V = \epsilon^{1/n} \left[V^{(0)} + \dots \right]. \tag{42}$$

Note that a term of order unity could be included in T_{xx} and T_{yy}, but from the required matching with the outer solution it is apparent that they vanish. Furthermore, note that the longitudinal stress S_{xx} within the inner region is of order $\epsilon^{1/n}$ and therefore it is large compared to its order ϵ counterpart in the outer region. Using the expansions (41) we now have

$$\tau^{n-1} = \epsilon^{1-1/n} \tau_0^{n-1} + \dots, \tag{43}$$

where

$$\tau_0^{n-1} = \left[S_{xx}^{(0)^2} + S_{xy}^{(0)^2} \right]^{(n-1)/2}$$

For the governing equations we find at leading order,

$$\frac{\partial T_{yy}^{(-1)}}{\partial y} = 0, \qquad T_{xx}^{(-1)} = T_{yy}^{(-1)}, \tag{44}$$

and at second order

$$\frac{\partial S_{xy}^{(0)}}{\partial Y} = -\frac{\partial T_{xx}^{(-1)}}{\partial x}, \qquad \frac{\partial T_{yy}^{(0)}}{\partial Y} = 0, \tag{45}$$

$$\frac{\partial U^{(0)}}{\partial Y} = 0, \qquad \frac{\partial U^{(0)}}{\partial x} = \tau_0^{n-1} S_{xx}^{(0)} = -\frac{\partial V^{(0)}}{\partial Y}. \tag{46}$$

where $T_{xx}^{(0)} = T_{yy}^{(0)} + 2S_{xx}^{(0)}$. Therefore $T_{xx}^{(-1)}$, $T_{yy}^{(-1)}$, $T_{yy}^{(0)}$ and $U^{(0)}$ do not vary across the inner region (i.e., they are independent of Y) and will be determined by matching with the outer solution. The matching gives

$$T_{xx}^{(-1)} = T_{yy}^{(-1)} = 2^{1+1/n}(n+2)^{1/n}\frac{n}{1+n}\left(x^{(1+n)/n} - 1\right), \tag{47}$$

$$T_{yy}^{(0)} = 0, \qquad U^{(0)} = 2\frac{n+2}{n+1}x.$$

From eqn. (45) we find

$$S_{xy}^{(0)} = -2^{1+1/n}(n+2)^{1/n}x^{1/n}Y,$$

and (46) gives the following equation for $S_{xx}^{(0)}$:

$$\left[S_{xx}^{(0)^2} + S_{xy}^{(0)^2}\right]^{(n-1)/2} S_{xx}^{(0)} = 2\frac{n+2}{n+1} \tag{48}$$

The asymptotic behavior of Eqn. (48) is readily examined. As $Y \to \infty$ we have

$$S_{xx}^{(0)} \to 2\frac{n+2}{n+1}\frac{1}{|S_{xy}^{(0)}|^{n-1}},$$

and therefore the solution matches with the inner limit of the outer solution. Furthermore, as $Y \to 0$ the longitudinal stress along the centerline is found to be

$$S_{xx} \simeq \epsilon^{1/n} S_{xx}^{(0)} \to \left(2\epsilon\frac{n+2}{n+1}\right)^{1/n}.$$

Recall that this was predicted to be infinite by the lubrication approximation.

For the special cases $n = 2$ and $n = 3$ an exact solution to (48) may be conveniently obtained as,

$$S_{xx}^{(0)} = \frac{1}{\sqrt{2}}\left[\sqrt{S_{xy}^{(0)^4} + \left(\frac{16}{3}\right)^2} - S_{xy}^{(0)^2}\right]^{1/2}, \qquad (n=2) \tag{49}$$

$$S_{xx}^{(0)} = \left[\tfrac{5}{4} - \sqrt{\left(\tfrac{1}{3}S_{xy}^{(0)^2}\right)^3 + \left(\tfrac{5}{4}\right)^2}\right]^{1/3} + \left[\tfrac{5}{4} + \sqrt{\left(\tfrac{1}{3}S_{xy}^{(0)^2}\right)^3 + \left(\tfrac{5}{4}\right)^2}\right]^{1/3} \qquad (n=3) \tag{50}$$

Note that in order to completely match through $O(\epsilon)$ in the outer solution it would also be necessary to obtain the stress field in the inner region through terms of $O(\epsilon)$. This, however, will not be pursued here.

5. Midpoint inner solution

Here we introduce the inner variable $X = x/\epsilon$, and again using capitals to denote the inner solution the governing equations become

$$\frac{\partial T_{xx}}{\partial X} + \frac{\partial S_{xy}}{\partial y} = 0, \qquad \frac{\partial S_{xy}}{\partial X} + \frac{\partial T_{yy}}{\partial y} = 0, \tag{51}$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial y} = 0, \tag{52}$$

$$\epsilon\frac{\partial U}{\partial X} = \tau^{n-1}S_{xx} = -\epsilon\frac{\partial V}{\partial y}, \qquad \epsilon\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial X}\right) = 2\tau^{n-1}S_{xy}. \tag{53}$$

For the inner expansion we take

$$T_{kk} = \epsilon^{-1}T_{kk}^{(-1)} + \epsilon^{1/n}T_{kk}^{(0)} + \dots, \qquad (k = x \text{ or } y, \text{ no sum})$$

$$S_{ij} = \epsilon^{1/n}S_{ij}^{(0)} + \dots, \tag{54}$$

$$U = U^{(0)} + \dots, \qquad V = V^{(0)} + \dots. \tag{55}$$

As before, an examination of the outer solution reveals that no terms of order unity are needed in T_{xx} and T_{yy}. Note that the two velocity components are of the same order of magnitude here. Furthermore, anticipating that the inner region does not substantially affect the velocity field, we assume that the velocity field is given by

$$\begin{aligned} U^{(0)} &= U^{(0)}(X, y) = -(2)^{n+2}\frac{n+2}{n+1}(X + n\alpha^{-1}C)\left[y^{n+1} - (\tfrac{1}{2})^{n+1}\right], \\ V^{(0)} &= V^{(0)}(y) = (2)^{n+2}\frac{n+2}{n+1}y\left[\frac{y^{n+1}}{n+2} - \left(\frac{1}{2}\right)^{n+1}\right]. \end{aligned} \tag{56}$$

This is the inner limit of the outer velocity field and exactly satisfies the continuity equation (52). Recall that $\alpha = 2^{1+1/n}(n+2)^{1/n}$ and that the constant C in (56) is the integration constant which was introduced in the function $G_0(x)$ earlier. The ability to construct the complete solution will ultimately verify that the velocity field is indeed given by (56).

The first two terms in the stress field have the governing equations,

$$\frac{\partial T_{xx}^{(-1)}}{\partial X} = 0, \qquad \frac{\partial T_{yy}^{(-1)}}{\partial y} = 0. \tag{57}$$

Furthermore $T_{xx}^{(-1)} = T_{yy}^{(-1)}$ and therefore after matching to the outer solution we find

$$T_{xx}^{(-1)} = T_{yy}^{(-1)} = -2^{1+1/n}(n+2)^{1/n}\frac{n}{1+n}. \tag{58}$$

At $O(\epsilon^{1/n})$ we have

$$\frac{\partial T_{xx}^{(0)}}{\partial X} + \frac{\partial S_{xy}^{(0)}}{\partial y} = 0, \qquad \frac{\partial S_{xy}^{(0)}}{\partial X} + \frac{\partial T_{yy}^{(0)}}{\partial y} = 0, \tag{59}$$

and since $V^{(0)}$ is a function only of y, the constitutive equations give

$$\tau_0^{n-1} S_{xx}^{(0)} = \frac{\partial U^{(0)}}{\partial X} = -2^{n+2}\frac{n+2}{n+1}\left[y^{n+1} - (1/2)^{n+1}\right] \equiv e_{xx}(y), \tag{60}$$

$$\tau_0^{n-1} S_{xy}^{(0)} = \frac{1}{2}\frac{\partial U^{(0)}}{\partial y} = -2^{n+1}(n+2)(X + n\alpha^{-1}C)y^n \equiv e_{xy}(X, y), \tag{61}$$

where $\tau_0^{n-1} = [S_{xx}^{(0)^2} + S_{xy}^{(0)^2}]^{(n-1)/2}$. From (60) and (61) we find

$$S_{xx}^{(0)} = \frac{e_{xx}}{\left(e_{xx}^2 + e_{xy}^2\right)^{(n-1)/2n}}, \qquad S_{xy}^{(0)} = \frac{e_{xy}}{\left(e_{xx}^2 + e_{xy}^2\right)^{(n-1)/2n}}, \tag{62}$$

where e_{xx} and e_{xy} are defined in (60) and (61). Now, from the symmetry condition $S_{xy}^{(0)}(X=0) = 0$ we must take $C = 0$.

Lastly, using $T_{xx}^{(0)} = T_{yy}^{(0)} + 2S_{xx}^{(0)}$ (59) becomes

$$\frac{\partial T_{yy}^{(0)}}{\partial X} = -\frac{\partial S_{xy}^{(0)}}{\partial y} - 2\frac{\partial S_{xx}^{(0)}}{\partial X} \equiv f(X, y), \tag{64}$$

$$\frac{\partial T_{yy}^{(0)}}{\partial y} = -\frac{\partial S_{xy}^{(0)}}{\partial X} \equiv g(X, y). \tag{65}$$

where f and g are easily determined from (60), (61), and (62). Equations (64) and (65) show that $T_{yy}^{(0)} =$ constant on curves given by

$$\frac{\mathrm{d}y}{\mathrm{d}X} = -\frac{f(X, y)}{g(X, y)} = -n\frac{X}{y} - 8\frac{n-1}{n+1}\frac{Xy^n\left(1/2^{n+1} - y^{n+1}\right)}{\frac{4n}{(n+1)^2}\left(1/2^{n+1} - y^{n+1}\right)^2 + X^2y^{2n}}. \tag{66}$$

These curves were obtained numerically for $n = 1$, 3 and 5 using a fourth-order Runge-Kutta scheme. The results are shown in Fig. 2. For the case $n = 1$, i.e., a Newtonian fluid, it is easily verified that the curves are arcs of a circle centered at the origin. The constant value of $T_{yy}^{(0)}$ on each curve is most easily determined by find the value of $T_{yy}^{(0)}$ where the

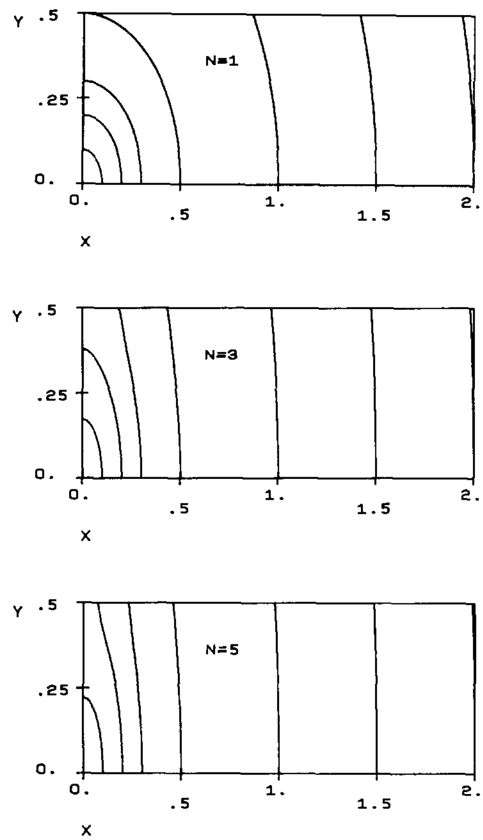

Figure 2. Curves in the midpoint inner region on which T_{yy} = constant: top, $n = 1$; middle, $n = 3$; bottom, $n = 5$. The X-axis is the centerline of the gap.

curve intercepts either the line $y = 1/2$ or the line $x = 0$. From (64), (65), and matching with the outer solution the stresses on $y = 1/2$ and $x = 0$ are respectively.

$$T_{yy}^{(0)}(X, 1/2) = 2^{1+1/n}(n+2)^{1/n}\frac{n}{1+n}X^{1+1/n}, \tag{67}$$

$$T_{yy}^{(0)}(0, y) = \left\{4\frac{n+2}{n+1}\left[\left(\tfrac{1}{2}\right)^{n+1} - y^{n+1}\right]\right\}^{1/n} \tag{68}$$

Note that matching with the outer solution is achieved since it is easily verified that S_{xx} and S_{xy} match as $X \to \infty$ and since the curves described by (66) asymptote to X = constant ($\mathrm{d}y/\mathrm{d}X \to \infty$) as $X \to \infty$ and therefore T_{yy} matches.

Acknowledgement

I would like to thank Robert Smet for this comments concerning the original manuscript. This work was supported, in part, by a grant from the National Science Foundation (MEA 81-07564).

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